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Two Questions of L. A. Shemetkov on Critical Groups

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1. INTRODUCTION

Throughout the paper we consider only finite groups.

Let $\mathfrak{X}$ be a class of groups. A group G is called s-critical for $\mathfrak{X}$, or simply $\mathfrak{X}$-critical, if G is not in $\mathfrak{X}$ but all proper subgroups of G are in $\mathfrak{X}$. Following Doerk and Hawkes [3, VII, (6.1)], we denote $\mathrm{Crit}_s(\mathfrak{X})$ the class of all $\mathfrak{X}$-critical groups. Knowledge of the structure of the groups in $\mathrm{Crit}_s(\mathfrak{X})$ for a class of groups $\mathfrak{X}$ can often help one to obtain detailed information for the structure of the groups belonging to $\mathfrak{X}$.

O. J. Schmidt (see [5, III, (5.2)]) studied the $\mathfrak{N}$-critical groups, where $\mathfrak{N}$ is the formation of the nilpotent groups. These groups are also called Schmidt groups. In [2], answering to a question posed by Shemetkov in the Kourovka Notebook [6, p. 84], the authors characterized those subgroup-closed saturated formations $\mathfrak{f}$ of finite groups such that every $\mathfrak{f}$-critical group is either a Schmidt group or a cyclic group of prime order. We shall say that a saturated formation $\mathfrak{f}$ has the Shemetkov property if every $\mathfrak{f}$-critical group is either a Schmidt group or a cyclic group of prime order. Since the structure of the Schmidt groups is well known, the structure of the $\mathfrak{f}$-critical groups, where $\mathfrak{f}$ is a saturated formation with the Shemetkov property, is determined as well.

Shemetkov [7, Problem 10.22] proposes the following question:

"Let $\mathfrak{f}$ be a non-empty subgroup-closed formation of finite groups. Assume that $\mathfrak{f}$ has the Shemetkov property. Is $\mathfrak{f}$ local?"

Skiba [8] answers this question affirmatively in the soluble universe. He proves that if $\mathfrak{f}$ is a non-empty subgroup-closed formation of soluble groups with the Shemetkov property, then $\mathfrak{f}$ is saturated or, equivalently, local.

We prove here that this result does not remain true in the general case (Example 2). In fact, in Theorem 2, we give a criterion for a subgroup-closed

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formation with the Shemetkov property to be saturated. Moreover we see that the subgroup-closed saturated formations with the Shemetkov property can be characterized as the subgroup-closed saturated formations for which an extension of the well known Frobenius p-nilpotence criterion holds (Theorem 1).

The second part of the paper is devoted to answering the following question posed by Shemetkov in the Kourovka Notebook [6, Problem 8.87]:

"Find all subgroup-closed saturated formations $\mathfrak{F}$ satisfying the following condition: every $\mathfrak{F}$-critical group is biprimary." (Recall that Schmidt groups are biprimary.)

Theorems 3 and 4 contain the answer to this question.

The notation agrees with that in the book of Doerk and Hawkes [3]. We refer the reader to that book for the basic notation, terminology, and results on formations.

2. PRELIMINARIES

In this section we collect some definitions and results that are needed in the sequel.

Recall that if $\mathfrak{F}$ is a saturated formation, there exists a unique formation function, F say, defining $\mathfrak{F}$ which is integrated and full (see [3, IV, (3.8)]); F is called the canonical local definition of $\mathfrak{F} = LF(F)$. If π is a set of primes, we denote by $\mathfrak{E}_\pi$ (respectively $\mathfrak{S}_\pi$) the class of all π-groups (respectively soluble π-groups).

If $\mathfrak{X}$ is a class of groups, we denote char $\mathfrak{X} = \{p : p$ is a prime number and $C_p \in \mathfrak{X}\}$. The boundary of a class $\mathfrak{X}$ is the class $b(\mathfrak{X}) = (G \in \mathfrak{E} : G \notin \mathfrak{X}$ and if $1 \neq N \trianglelefteq G$ then $G/N \in \mathfrak{X})$. For a group G, $\pi(G)$ is the set of all prime divisors of its order. A group G is said to be biprimary if $|\pi(G)| \leq 2$. We denote by $\mathscr{P}$ the class of all biprimary groups. If p is a prime, $\mathscr{P}(p)$ is the class of all biprimary groups G of order $p^a q^b$ for some prime q and $a, b \geq 0$.

The following result is proved in [2, Theorems 1, 2].

2.1. THEOREM. (a) *Let $\mathfrak{F} = LF(F)$ be a subgroup-closed saturated formation. Assume that $\mathfrak{F}$ has the Shemetkov property. Then, for each prime $p \in$ char $\mathfrak{F} = \pi$, we have $F(p) = \mathfrak{E}_{\pi(p)} \cap \mathfrak{F}$, where $\pi(p) =$ char $F(p)$. In particular, for every prime number $p \in \pi$, there exists a set of primes $\pi(p)$ with $p \in \pi(p)$ such that $\mathfrak{F}$ is locally defined by the formation function f given by $f(p) = \mathfrak{E}_{\pi(p)}$ if $p \in \pi$, and $f(q) = \emptyset$ if $q \notin \pi$.*

(b) *Let $\mathfrak{F}$ be a subgroup-closed saturated formation. Then $\mathfrak{F}$ has the Shemetkov property if and only if $\mathfrak{F}$ satisfies the following two conditions*:

(i) *There exists a formation function f as in* (a) *such that* $\mathfrak{f} = LF(f)$; *this formation function f satisfies the following property*:

$$\begin{array}{ll}\text{If } G \in \mathrm{Crit}_s(\mathfrak{f}) \cap b(\mathfrak{f}) \text{ and } G \text{ is an almost simple group} & \\ \text{such that } G \notin f(p) \text{ for some prime } p \in \pi(\mathrm{Soc}(G)), & (*) \\ \text{then } G \notin f(q) \text{ for each prime } q \in \pi(\mathrm{Soc}(G)). & \end{array}$$

(ii) $\mathrm{Crit}_s(\mathfrak{f}) \cap b(\mathfrak{f})$ *does not contain non-abelian simple groups.*

Let K be a field of characteristic p (p a prime number) and let G be a group. Consider K as a trivial KG-module and denote by P_K the indecomposable projective KG-module with K in the head, i.e., $P_K/P_K J \cong K$, where J is the Jacobson radical of KG (we consider only finite modules). $A_K(G)$ denotes the kernel of a projective cover $P_1 \twoheadrightarrow P_K J$; $A_K(G)$ is uniquely determined up to isomorphism.

2.2. THEOREM (W. Gaschütz; see [4]). *Let* $K = GF(p)$, *the finite field of p elements, and* $A = A_K(G)$. *Then there exists a Frattini extension* $A \rightarrowtail G^* \twoheadrightarrow G$, *i.e., with* $A \leq \Phi(G^*)$. *Any other Frattini extension of G by a KG-module is an epimorphic image over G of* G^*.

$A_K(G)$ is called the Frattini module of G with respect to K. It is known that $A_K(G) \neq 0$ if p divides the order of G; for a p'-group G, $A_K(G) = 0$.

Let G be a Schmidt group. It is known that $\pi(G) = \{p, q\}$, for two distinct primes p and q. G has a normal Sylow p-subgroup P whereas the Sylow q-subgroups of G are cyclic. In this case we say that G is of type (p, q). If the Frattini subgroup of G is trivial, then P is an elementary abelian minimal normal p-subgroup of G and so P is an irreducible and faithful Q-module over $GF(p)$, where Q is a Sylow q-subgroup of G of order q. In this case G is isomorphic to the group $E(q \mid p)$ constructed in [3, B, (12.4)].

3. THE RESULTS

THEOREM 1. *Let* $\mathfrak{f} = LF(F)$ *be a subgroup-closed saturated formation. Denote* $\pi = \mathrm{char}\ \mathfrak{f}$ *and* $\pi(p) = \mathrm{char}\ F(p)$, *for every* $p \in \pi$. *The following statements are pairwise equivalent*:

(i) $\mathfrak{f}$ *has the Shemetkov property.*

(ii) *A* π*-group G belongs to* $\mathfrak{f}$ *if and only if* $N_G(Q)/C_G(Q)$ *belongs to* $\mathfrak{E}_{\pi(p)}$ *for each p-subgroup Q of G and each prime* $p \in \pi$.

(iii) *A* π*-group G belongs to* $\mathfrak{f}$ *if and only if* $N_G(Q)$ *belongs to* $\mathfrak{E}_{\pi\text{-}\{p\}}\mathfrak{E}_{\pi(p)}$ *for each p-subgroup Q of G and each prime* $p \in \pi$.

Proof. (i) implies (ii). Assume that $\mathfrak{f}$ has the Shemetkov property. According to Theorem 2.1, we have $F(p) = \mathfrak{E}_{\pi(p)} \cap \mathfrak{f}$, for every $p \in \pi$. Let G be a π-group in $\mathfrak{f}$. Suppose that a prime $p \in \pi$ is fixed and let Q

be a p-subgroup of G. Then $N_G(Q)$ belongs to $\mathfrak{f}$ because $\mathfrak{f}$ is subgroup-closed. In particular, $N_G(Q)/O_{p'}(N_G(Q)) \in F(p) \subseteq \mathfrak{E}_{\pi(p)}$. Since Q is a normal p-subgroup of $N_G(Q)$, it follows that $O_{p'}(N_G(Q)) \leq C_G(Q)$. This means that $N_G(Q)/C_G(Q)$ belongs to $\mathfrak{E}_{\pi(p)}$. Conversely, assume that G is a π-group such that $N_G(Q)/C_G(Q)$ belongs to $\mathfrak{E}_{\pi(p)}$ for each p-subgroup Q of G and each $p \in \pi$, but G is not an $\mathfrak{f}$-group. If we choose G of minimal order among the groups $X \notin \mathfrak{f}$ satisfying the above property, we have that G is an $\mathfrak{f}$-critical group because this property holds in every subgroup of G. Since $\mathfrak{f}$ has the Shemetkov property, it follows that G is a Schmidt group. In particular $\pi(G) = \{p, q\}$, for two distinct primes p and q in π and G has a normal Sylow p-subgroup, P say. By hypothesis, we have that $G/C_G(P)$ belongs to $\mathfrak{E}_{\pi(p)}$. If q were not in $\pi(p)$, it would be true that $Q \leq C_G(P)$, a contradiction. So $q \in \pi(p)$ and then $Q \in \mathfrak{E}_{\pi(p)} \cap \mathfrak{f} = F(p)$. Therefore $G \in \mathfrak{S}_p F(p) = F(p) \subseteq \mathfrak{f}$, a contradiction.

(ii) implies (i). We see that $\mathfrak{f}$ satisfies the part (b) of Theorem 2.1.

(a) For each prime $p \in \operatorname{char} \mathfrak{f} = \pi$, we have $F(p) = \mathfrak{E}_{\pi(p)} \cap \mathfrak{f}$. Let p be a prime in π. Since $F(p)$ is subgroup-closed by [3, IV, (3.16)], we have that $F(p)$ is contained in $\mathfrak{E}_{\pi(p)} \cap \mathfrak{f}$. Assume that $F(p)$ is not equal to $\mathfrak{E}_{\pi(p)} \cap \mathfrak{f}$ and take a group $G \in (\mathfrak{E}_{\pi(p)} \cap \mathfrak{f}) \setminus F(p)$ of minimal order. Then $1 \neq \operatorname{Soc}(G)$ is the unique minimal normal subgroup of G and it is not a p-group. By [3, B, (10.7), (10.9)] there exists an irreducible and faithful G-module V over $GF(p)$. Let $X = [V]G$ be the corresponding semidirect product. It is clear that X is a primitive group and X is not an $\mathfrak{f}$-group because G is not an $F(p)$-group. Moreover X is an $\mathfrak{f}$-critical group (see the proof of Theorem 2.1 in [2]). Let q be a prime in π and let Q be a q-subgroup of X. Suppose that $p \neq q$. Then $N_X(Q)$ is a proper subgroup of X because V is the unique minimal normal subgroup of X. This implies that $N_X(Q) \in \mathfrak{f}$ because X is $\mathfrak{f}$-critical. In particular $N_X(Q) \in \mathfrak{E}_{\pi\text{-}\{q\}}\mathfrak{E}_{\pi(q)}$ and so $N_X(Q)/C_X(Q) \in \mathfrak{E}_{\pi(q)}$. Now, if $p = q$ and $N_X(Q)$ is a proper subgroup of X, we can argue as above to conclude that $N_X(Q)/C_X(Q) \in \mathfrak{E}_{\pi(p)}$. If Q is a normal subgroup of X, then $V = Q$ by a known property of the primitive groups. In this case we also have $X/C_X(Q) \in \mathfrak{E}_{\pi(p)}$ because $X/C_X(Q)$ is isomorphic to G. Since X is a π-group, we can apply (ii) to conclude that X is an $\mathfrak{f}$-group, a contradiction. Therefore (a) holds.

(b) $\operatorname{Crit}_s(\mathfrak{f}) \cap b(\mathfrak{f})$ does not contain primitive groups of type 2. Assume that G is a primitive group of type 2 in $\operatorname{Crit}_s(\mathfrak{f}) \cap b(\mathfrak{f})$. Since G is $\mathfrak{f}$-critical, we have that G is a π-group. On the other hand, applying (ii), we can determine a prime $p \in \pi$ and a p-subgroup Q of G such that $N_G(Q)/C_G(Q)$ does not belong to $\mathfrak{E}_{\pi(p)}$. Suppose that $N_G(Q)$ is a proper subgroup of G. Then $N_G(Q) \in \mathfrak{f} \subseteq \mathfrak{E}_{\pi\text{-}\{p\}}\mathfrak{E}_{\pi(p)}$. This means that $N_G(Q)/C_G(Q) \in \mathfrak{E}_{\pi(p)}$, a contradiction. So Q is a normal subgroup of G.

But then $\mathrm{Soc}(G) \leq Q$ and $\mathrm{Soc}(G)$ is abelian, a contradiction. Consequently (b) holds.

From (a) and (b) we deduce that $\mathfrak{f}$ has the properties given in Theorem 2.1. This means that $\mathfrak{f}$ has the Shemetkov property.

Assume now that G is a π-group. Let p be a prime in π and let Q be a p-subgroup of G. If $N_G(Q) \in \mathfrak{E}_{\pi\text{-}\{p\}}\mathfrak{E}_{\pi(p)}$, then $N_G(Q)/C_G(Q) \in \mathfrak{E}_{\pi(p)}$. This elementary remark proves that (ii) implies (iii). Now, if (iii) holds, we can repeat the arguments used in the proof of (ii) implies (i) to conclude that $\mathfrak{f}$ has the Shemetkov property. Consequently (iii) implies (i) and the circle of implications is closed.

Example 1. Let $\mathfrak{f} = \mathfrak{E}_{p'}\mathfrak{S}_p$ be the subgroup-closed saturated formation of p-nilpotent groups, p a prime number. It is clear that $\mathfrak{f} = LF(F)$, where $F(p) = \mathfrak{S}_p$ and $F(q) = \mathfrak{f}$ for every prime $q \neq p$. Moreover, $\mathfrak{f}$ has the Shemetkov property (see [2, Example 4]). Therefore $\mathfrak{f}$ satisfies the condition (ii) of Theorem 1. Hence a group G is p-nilpotent if and only if $N_G(Q)/C_G(Q)$ is a p-group for every p-subgroup Q of G. The statement (iii) of this theorem says that a group G is p-nilpotent if and only if $N_G(Q)$ is p-nilpotent for every p-subgroup Q of G. These statements are two equivalent forms of the well known p-nilpotence criterion due to Frobenius.

As we said in the Introduction, Skiba [8], answering to a question posed by Semenchuk, proved that a formation of soluble groups with the Shemetkov property in the soluble universe should be saturated, or equivalently, local.

This result does not hold in the general universe of all finite groups, as the next example shows:

Example 2. Consider the class $\mathfrak{f} = (G \in \mathfrak{E} : \text{each } \{3, 5\}\text{-subgroup of } G$ is nilpotent). According to [3, VII, (6.5)], $\mathfrak{f}$ is a subgroup-closed formation.

$\mathfrak{f}$ has the Shemetkov property: For, let G be an $\mathfrak{f}$-critical group. Then there exists a $\{3, 5\}$-subgroup H of G which is not nilpotent. By the choice of G, it follows that $G = H$. Especially, G is a $\{3, 5\}$-group which is not nilpotent but all its subgroups are nilpotent. Therefore G is a Schmidt group.

$\mathfrak{f}$ is not a saturated formation: Take the group $G = A_5$, the alternating group of degree 5. Let E be the maximal Frattini extension of G with respect to the prime 5. Then $E/\Phi(E)$ is isomorphic to G and $C_G(\Phi(E)) = O_{5'}(G) = 1$ by [4, Proposition 5]. It is clear that $G \in \mathfrak{f}$. If the formation $\mathfrak{f}$ were saturated, it would be true that $E \in \mathfrak{f}$. But this is not true because $\Phi(E)P$, for a Sylow 3-subgroup P of E, is not nilpotent inasmuch as P does not centralize $\Phi(E)$.

Consequently $\mathfrak{f}$ is a subgroup-closed non-saturated formation with the Shemetkov property.

The following theorem provides a criterion for a subgroup-closed formation with the Shemetkov property to be saturated.

THEOREM 2. *Let $\mathfrak{f}$ be a subgroup-closed formation with the Shemetkov property. The following statements are equivalent*:

(i) *$\mathfrak{f}$ is a saturated formation.*

(ii) *Let G be a primitive group of type 2 such that $G \in \mathfrak{f}$. If p divides the order of $\mathrm{Soc}(G)$ and V is an irreducible and faithful G-module over $GF(p)$, then every Schmidt subgroup of type (p, q) of the semidirect product $[V]G$ belongs to $\mathfrak{f}$.*

Proof. If $\mathfrak{f}$ is a subgroup-closed saturated formation, then the statement (ii) is always true: Let $G \in \mathfrak{f}$ be a primitive group of type 2; then $G \in F(p)$ for every prime p dividing the order of $\mathrm{Soc}(G)$, where F is the canonical local definition of $\mathfrak{f}$. The semidirect product $[V]G \in \mathfrak{S}_p F(p) = F(p) \subseteq \mathfrak{f}$, for each irreducible and faithful G-module V over $GF(p)$ and p dividing the order of $\mathrm{Soc}(G)$. Now the result is clear because $\mathfrak{f}$ is subgroup-closed.

(ii) implies (i). Assume that (ii) holds but $\mathfrak{f}$ is not saturated. Let G be a group of minimal order in $E_\Phi\mathfrak{f} \setminus \mathfrak{f}$. Then G has a unique minimal normal subgroup N such that $N \leq \Phi(G)$, in particular N is abelian, and $G/N \in \mathfrak{f}$. Let p be the prime dividing $|N|$. Since $G \notin \mathfrak{f}$, there exists a subgroup H of G such that H is a Schmidt group and $H \notin \mathfrak{f}$. Suppose that H is a proper subgroup of G. Since $HN/N \in \mathfrak{f}$ and $H \notin \mathfrak{f}$, it follows that $H \cap N \neq 1$. If $H \cap N$ were contained in $\Phi(H)$, then $H \in E_\Phi\mathfrak{f}$. This would imply that $H \in \mathfrak{f}$ by minimality of G, a contradiction. Therefore $H \cap N$ is not contained in $\Phi(H)$ and then H is a Schmidt group of type (p, q) for some prime $q \neq p$. Since q divides the order of G/N, we can find an element g of G such that the order of gN is q. Since $O_{p'p}(G/N) = O_{p'p}(G)/N = O_p(G)/N$ is the intersection of the centralizers of the chief factors of G/N whose order is divisible by p and gN is a q-element, it follows that there exists a chief factor L/K of G whose order is divisible by p such that $N \leq K < L$ and $g \notin C_G(L/K)$. Suppose that L/K is non-abelian. Then $G^* = G/C_G(L/K) \in \mathfrak{f}$ is a primitive group of type 2 whose socle is G-isomorphic to L/K. In particular p divides the order of $\mathrm{Soc}(G^*)$. According to [3, B, (10.9)] there exists an irreducible and faithful G^*-module over $GF(p)$. Let $X = [V]G^*$ be the corresponding semidirect product. Denote $A = \langle gC_G(L/K)\rangle$ and consider the subgroup VA of X. V, regarded as an A-module, is semisimple by virtue of a well known result of Maschke. Moreover since V is faithful and A is a cyclic group of order

q, we can find an irreducible A-submodule W of V such that W is a faithful A-module. Let $B = WA$ be the corresponding semidirect product. It is clear that B is isomorphic to $E(q \mid p)$. By (ii), the group B is in $\mathfrak{f}$. If L/K is abelian, then $Z = [L/K]G^* \in \mathfrak{f}$ by a result of Barnes and Kegel (see [3, IV, (1.5)]). Since L/K is an irreducible and faithful G^*-module over $GF(p)$, we can argue as above and conclude that there exists a subgroup D of Z such that D is isomorphic to $E(q \mid p)$ and $D \in \mathfrak{f}$. Therefore, in both cases, we can find a Schmidt group of the type $E(q \mid p)$ in the formation $\mathfrak{f}$.

Assume that $\Phi(H) \neq 1$. Then $H/\Phi(H)$ is isomorphic to the group $E(q \mid p)$ constructed above. This implies that $H \in E_\Phi \mathfrak{f}$ and so $H \in \mathfrak{f}$ by minimality of G, a contradiction. Hence $\Phi(H) = 1$. But then H is isomorphic to $E(q \mid p) \in \mathfrak{f}$, a contradiction. Consequently, $H = G$ and G is a Schmidt group of type (p, q). In particular, $\Phi(P)$ is an elementary abelian group, if $P \in \mathrm{Syl}_p(G)$. Let Q be a Sylow q-subgroup of G. Since N is the unique minimal normal subgroup of G and $\Phi(Q)$ is a normal subgroup of G, we have that $\Phi(Q) = 1$ and Q is a cyclic group of order q. Note that $G/\Phi(P) \cong E(q/p)$ and $\Phi(P) = \Phi(G)$. Then G is an epimorphic image of the maximal Frattini extension E of the group $E(q/p)$ with respect to the prime p. Denote by A the Frattini module of $E(q/p)$ with respect to $GF(p)$. Let Q^* be a Sylow q-subgroup of E. If $AQ^* \in \mathfrak{f}$, then E is $\mathfrak{f}$-critical because $G \notin \mathfrak{f}$ and A is contained in each maximal subgroup of E. This implies that E is a Schmidt group. In particular, AQ^* is nilpotent and then $1 \neq Q^* \leq C_E(\mathrm{Soc}(A))$ which is a p-group by [4, Theorem 1], a contradiction. Hence AQ^* is not an $\mathfrak{f}$-group. Again we can find a subgroup J of AQ^* such that J is a Schmidt group of type (p, q) and $J \notin \mathfrak{f}$. Moreover $A \cap J \in \mathrm{Syl}_p(J)$ is an elementary abelian group, which implies that J is isomorphic to $E(q/p) \in \mathfrak{f}$, final contradiction.

The final part of the paper is devoted to studying the formations $\mathfrak{f}$ with the following property:

(α) If G is an $\mathfrak{f}$-critical group, then G is biprimary.

It is clear that if $\mathfrak{f}$ has the Shemetkov property, then $\mathfrak{f}$ has the property (α) but the converse is not true in general (see Example 3).

A biprimary formation function is a function f which associates with each pair of primes $\{p, q\}$ a formation $f(p, q) = f(q, p)$ of finite groups.

For a pair of primes $\{p, q\}$, denote by $S_{\{p,q\}}(G)$ the set of all $\{p, q\}$-subgroups of a group G.

THEOREM 3. *Let $\mathfrak{f}$ be a class of groups. The following statements are pairwise equivalent*:

(i) $\mathfrak{f}$ *is a subgroup-closed formation satisfying the property* (α).

(ii) $\mathfrak{f}$ *is a subgroup-closed formation such that* $\mathfrak{f} = (G \in \mathfrak{E}/S_{\{p,q\}}(G) \subseteq \mathfrak{f}$ *for all pairs of primes* $\{p, q\})$.

(iii) *There exists a biprimary formation function f such that* $\mathfrak{f} = (G \in \mathfrak{E}/S_{\{p,q\}}(G) \subseteq f(p, q))$.

Proof. If $\mathfrak{f}$ is a subgroup-closed formation, the equivalence between (i) and (ii) follows from [3, VII, (6.2)]. Assume now that $\mathfrak{f}$ is a subgroup-closed formation satisfying the property (α) and define $f(p, q) = \mathfrak{f}$ for every pair of primes $\{p, q\}$. It is clear that f is a biprimary formation function satisfying (iii). So (i) implies (iii).

Finally, let f be a biprimary formation function defining $\mathfrak{f}$ as in (iii). According to [3, VII, (6.5)], $\mathfrak{f}$ is certainly a subgroup-closed formation. Moreover, if G is an $\mathfrak{f}$-critical group, there exist a pair of primes $\{p, q\}$ and a $\{p, q\}$-subgroup H of G such that $H \notin f(p, q)$. Suppose that H is a proper subgroup of G. Then H belongs to $\mathfrak{f}$ and so $H \in f(p, q)$, a contradiction. Therefore we must have $G = H$. Especially, G is biprimary and $\mathfrak{f}$ has the property (α). Consequently (iii) implies (i) and the circle of implications is complete.

THEOREM 4. *Let* $\mathfrak{f}$ *be a saturated formation with* char $\mathfrak{f} = \pi$. *The following statements are pairwise equivalent*:

(i) $\mathfrak{f}$ *is subgroup-closed and has the property* (α).

(ii) (a) $\mathfrak{f}$ *is subgroup-closed.*

(b) $\mathfrak{f} \cap \mathrm{Crit}_{\mathrm{s}}(F(p))$ *is contained in* $\mathscr{P}(p)$, *for each prime* $p \in \pi$.

(c) *If G is a primitive group of type* 2 *in* $\mathrm{Crit}_{\mathrm{s}}(\mathfrak{f}) \cap b(\mathfrak{f})$, *then G is not simple and for each prime* $p \in \pi(\mathrm{Soc}(G))$, *there is no core-free maximal subgroup of G in* $F(p)$.

(iii) (a) $\mathfrak{f}$ *is locally defined by a formation function f given by* $f(p) = (G \in \mathfrak{E} : S_{\{p,q\}}(G) \subseteq f(p, q)$, *where* $f(p, q)$ *is a formation such that* $\mathfrak{S}_p f(p, q) = f(p, q)$ *for all* $q)$ *for each prime* $p \in \pi$, *and* $f(r) = \emptyset$ *if* $r \notin \pi$.

(b) $\mathfrak{f}$ *satisfies* (ii)(c).

Proof. (i) implies (ii). Assume that $\mathfrak{f}$ has the property (α). Since $\mathscr{P}$ is a class composed of soluble groups and $\mathrm{Crit}_{\mathrm{s}}(\mathfrak{f})$ is contained in $\mathscr{P}$, it follows that $\mathrm{Crit}_{\mathrm{s}}(\mathfrak{f})$ does not contain primitive groups of type 2. In particular, the third statement of (ii) holds.

Suppose now that a prime $p \in$ char $\mathfrak{f}$ is fixed. We see that $\mathfrak{f} \cap \mathrm{Crit}_{\mathrm{s}}(F(p))$ is contained in $\mathscr{P}(p)$. For each prime q we define $H(q) = (G \in \mathfrak{E}/S_{\{p,q\}}(G) \subseteq F(p))$. According to [3, VII, (6.5)], $H(q)$ is a subgroup-closed formation. Denote $\mathfrak{h} = \bigcap\{H(q)/q$ a prime number$\}$. It is clear that $\mathfrak{h}$ is a subgroup-closed formation. We prove that $\mathfrak{f} \cap \mathfrak{h} = F(p)$. First of all $F(p) \subseteq \mathfrak{f} \cap \mathfrak{h}$ because $F(p)$ is subgroup-closed and $F(p) \subseteq \mathfrak{f}$. Suppose that $F(p)$ is not equal to $\mathfrak{h} \cap \mathfrak{f}$ and take a group $G \in (\mathfrak{h} \cap \mathfrak{f}) \setminus F(p)$ of minimal order. Then $1 \neq \mathrm{Soc}(G)$ is the unique minimal normal subgroup of G and it is not a p-group. By [3, B, (10.7) and (10.9)] there exists an irreducible and faithful G-module V over $GF(p)$. Let $X = [V]G$ be the corresponding semidirect product. It is clear that X is a primitive group and X is not an $\mathfrak{f}$-group because G is not an $F(p)$-group. Moreover X is an $\mathfrak{f}$-critical group (see the proof of Theorem 2.1 in [2]). Since $\mathfrak{f}$ has the property (α) and p divides the order of X, it follows that X is a $\{p, q\}$-group for some prime q and then G belongs to $F(p)$ because $G \in H(q)$, a contradiction. Hence $\mathfrak{f} \cap \mathfrak{h} = F(p)$.

Let G be a group in $\mathfrak{f} \cap \mathrm{Crit}_s(F(p))$. Then $G \notin \mathfrak{h}$ and so there exists a prime q such that $G \notin H(q)$. This means that $S_{\{p,q\}}(G)$ is not contained in $F(p)$. Since G is $F(p)$-critical, it follows that G is a $\{p, q\}$-group. That is, $G \in \mathscr{P}(p)$. Consequently (i) implies (ii).

(ii) implies (i). Suppose that $\mathrm{Crit}_s(\mathfrak{f})$ is not contained in $\mathscr{P}$ and let G be a group of minimal order in $\mathrm{Crit}_s(\mathfrak{f}) \setminus \mathscr{P}$. Then $\Phi(G) = 1$ and $G \in b(\mathfrak{f})$. In particular G is a primitive group.

We distinguish two cases:

Case 1. G is a primitive group of type 1. In this case $N = \mathrm{Soc}(G)$ is an abelian minimal normal subgroup of G. Let p be the prime dividing the order of N and let M be a core-free maximal subgroup of G such that $G = MN$ and $M \cap N = 1$. If $M \in F(p)$, then $G \in \mathfrak{S}_p F(p) = F(p)$, a contradiction. Hence M is not an $F(p)$-group. Let A be a maximal subgroup of M. It is clear that NA is a maximal subgroup of G and so $NA \in \mathfrak{f}$ because G is $\mathfrak{f}$-critical. Since $NA \in \mathfrak{S}_{p'} F(p)$ and $C_G(N) = N$, we have $A \in F(p)$. This means that $M \in \mathrm{Crit}_s(F(p))$. Moreover $M \in \mathfrak{f}$. Then $M \in \mathscr{P}(p)$ by (ii). This implies that $G \in \mathscr{P}(p)$ and G is biprimary, a contradiction.

Case 2. G is a primitive group of type 2. We have that $N = \mathrm{Soc}(G)$ is a non-abelian minimal normal subgroup of G. If $N = G$, then G is a non-abelian simple group. This contradicts (ii). So N is a proper subgroup of G. This implies that $N \in \mathfrak{f}$ because G is $\mathfrak{f}$-critical. Hence $\pi(N) \subseteq \mathrm{char}\, \mathfrak{f}$. Let M be a core-free maximal subgroup of G. With arguments similar to those used in Case 1 and using $C_G(N) = 1$, it is not difficult to prove that $M \in \mathfrak{f} \cap \mathrm{Crit}_s(F(p))$ for each prime $p \in \pi(N)$. By (ii), we have $|\pi(M)| \leq 2$. Moreover, since N is not soluble, we know that $\pi(N)$ is a set with at least three elements. This implies that $|\pi(M)| = 1$ and M is a q-group for some prime q.

Let P be a Sylow p-subgroup of N for some prime $p \in \pi(N)$. Then, by the Frattini argument, $G = NN_G(P)$. Moreover $N_G(P)$ is a proper subgroup of G. Let $A(p)$ be a maximal subgroup of G such that $N_G(P) \leq$

$A(p)$. Then $A(p)$ is a core-free maximal subgroup of G and so M is a p-group. If we take a prime $q \neq p$ such that $q \in \pi(N)$, we can repeat the above arguments to conclude that there exists a core-free maximal subgroup of G, $A(q)$ say, which is a q-group. Then $G = NA(p) = NA(q)$. Consequently $G/N \in \mathfrak{S}_p \cap \mathfrak{S}_q = (1)$ and $G = N$, a contradiction.

(i) implies (iii). It is enough to prove the statement (a). Let $f(p) = (G \in \mathfrak{E} : S_{\{p,q\}}(G) \subseteq F(p))$ if $p \in \pi$, and $f(r) = \emptyset$ if $r \notin \pi$. By [3, VII, (6.5)] it follows that f is a formation function such that $f(p)$ is subgroup-closed for every prime $p \in \pi$. We claim that $\mathfrak{f} = LF(f)$. Since $\mathfrak{f}$ is subgroup-closed, we have that $F(p)$ is subgroup-closed for all $p \in \pi$. This implies that $F(p) \subseteq f(p)$, for all $p \in \pi$. Therefore $\mathfrak{f} = LF(F) \subseteq LF(f)$. Assume there exists a prime $p \in \pi$ such that $F(p)$ is not equal to $f(p) \cap \mathfrak{f}$ and take a group $G \in (f(p) \cap \mathfrak{f}) \setminus F(p)$ of minimal order. Then $G \in \mathfrak{f} \cap \mathrm{Crit}_{\mathrm{s}}(F(p)) \cap f(p) \subseteq \mathscr{P}(p) \cap f(p)$ by virtue of (ii). This means that $G \in F(p)$, a contradiction. Consequently $F(p) = f(p) \cap \mathfrak{f}$ for all primes $p \in \pi$. Assume now that $\mathfrak{f} \neq LF(f)$ and take a group $G \in LF(f) \setminus \mathfrak{f}$. Then G is a primitive group in $\mathrm{Crit}_{\mathrm{s}}(\mathfrak{f})$ because $LF(f)$ is subgroup-closed. By hypothesis (i), we have that G is a primitive group of type 1. Let p be the prime dividing the order of $\mathrm{Soc}(G)$. If M is a core-free maximal group of G, then $M \in f(p) \cap \mathfrak{f} = F(p)$ and $G \in \mathfrak{f}$, a contradiction.

(iii) implies (ii). Again from [3, VII, (6.5)] we know that the function f defined in (iii)(a) is a formation function and clearly $f(p)$ is subgroup-closed, for every prime p. This implies that $\mathfrak{f}$ is a subgroup-closed saturated formation.

On the other hand, it is not difficult to prove that $\mathfrak{S}_p f(p) = f(p)$, for every prime p. Henceforth $F(p) = \mathfrak{f} \cap f(p)$, for every prime p. Now $\mathfrak{f} \cap \mathrm{Crit}_{\mathrm{s}}(F(p)) = \mathfrak{f} \cap \mathrm{Crit}_{\mathrm{s}}(f(p) \cap \mathfrak{f}) \subseteq \mathrm{Crit}_{\mathrm{s}}(f(p)) \subseteq \mathscr{P}(p)$ for every prime $p \in \pi$.

REMARKS ON THEOREM 4. (1) *In order to augment the knowledge about the formation functions in* (iii)(a), *notice that for a class of groups* $\mathfrak{h}$ *and a prime p the following statements are pairwise equivalent*:

(i) $\mathfrak{h}$ *is a subgroup-closed formation and satisfies the condition that* $\mathrm{Crit}_{\mathrm{s}}(\mathfrak{h})$ *be contained in* $\mathscr{P}(p)$.

(ii) $\mathfrak{h}$ *is a subgroup-closed formation such that* $\mathfrak{h} = (G \in \mathfrak{E} : S_{\{p,q\}}(G) \subseteq \mathfrak{h}$ *for every prime* $q)$.

(iii) $\mathfrak{h} = (G \in \mathfrak{E} : S_{\{p,q\}}(G) \subseteq h(p,q)$, $h(p,q)$ *a formation for each prime* $q)$.

(2) *The statement* (iii)(a) *implies* (ii)(a, b), *but the converse is not true*.

(3) *The hypothesis* $\mathfrak{S}_p f(p, q) = f(p, q)$ *for each pair of primes* $\{p, q\}$, *in* (iii)(a) *cannot be dispensed with in order to prove the theorem.*

(4) *In the statement* (ii)(c) *it is not enough to require that* $\mathrm{Crit_s}(\mathfrak{f}) \cap b(\mathfrak{f})$ *not contain non-abelian simple groups in order to prove the theorem.*

Proof. (2) The implication (iii)(a) then (ii)(a, b) was obtained in the proof of the theorem. For the converse, consider the subgroup-closed saturated formation of soluble groups $\mathfrak{f}$ locally defined by the formation function given by $g(2) = g(3) = g(5) = \mathfrak{S}_{\{2,3,5\}}$, $g(7) = \mathfrak{S}_7$, and $g(p) = \emptyset$ if $p \notin \{2, 3, 5, 7\}$. It is not difficult to see that $F(p) = g(p)$ for each prime p, and $\mathfrak{f} \cap \mathrm{Crit_s}(F(p)) \subseteq \mathscr{P}(p)$ for each prime $p \in \pi$. Assume that $\mathfrak{f} = LF(f)$, where f is a formation function as in (iii)(a). In particular, $\mathrm{Crit_s}(f(p)) \subseteq \mathscr{P}(p) \subseteq \mathfrak{S}$ and $\mathfrak{S}_p f(p) = f(p)$, which implies that $F(p) = f(p) \cap \mathfrak{f}$, for each prime $p \in \{2, 3, 5, 7\}$.

If $A_5 \notin f(p)$, for some $p \in \{2, 3, 5\}$, then $A_5 \in \mathrm{Crit_s}(f(p))$ and this is not possible. Therefore $A_5 \in f(p)$, for each $p \in \{2, 3, 5\}$, which implies that $A_5 \in \mathfrak{f}$, a contradiction.

(3) Let p, q, r be three different primes and f the formation function given by $f(p) = \mathfrak{S}_{\{q,r\}}\mathfrak{S}_p$, $f(q) = \mathfrak{S}_{\{p,r\}}\mathfrak{S}_q$, $f(r) = \mathfrak{S}_{\{p,q\}}\mathfrak{S}_r$, and $f(t) = \emptyset$ if $t \notin \{p, q, r\}$.

If $u \in \{p, q, r\}$, note that $\mathrm{Crit_s}(f(u)) \subseteq \mathrm{Crit_s}(\mathfrak{S}_{u'}\mathfrak{S}_u) \cup (C_v : v \neq p, q, r) \subseteq \mathscr{P}(u)$, because from [2] the groups in $\mathrm{Crit_s}(\mathfrak{S}_{u'}\mathfrak{S}_u)$ are biprimary. Therefore $f(u)$ can be defined as in (iii)(a) but for some prime v, $\mathfrak{S}_u f(u, v) \neq f(u, v)$ because $\mathfrak{S}_u f(u) \neq f(u)$.

Let $\mathfrak{f} = LF(f)$ be the saturated formation locally defined by f. Then $\mathrm{char}\ \mathfrak{f} = \{p, q, r\}$ and $F(p) = \mathfrak{S}_p\mathfrak{S}_{\{q,r\}}\mathfrak{S}_p \cap \mathfrak{f}$.

Now, let V_p be an irreducible and faithful C_r-module over $GF(p)$ and $H = [V_p]C_r$ the corresponding semidirect product. If V_q is an irreducible and faithful H-module over $GF(q)$, note that the semidirect product $[V_q]H$ belongs to $\mathrm{Crit_s}(F(p)) \cap \mathfrak{f}$ but is not biprimary.

(4) Let G be a non-simple primitive group of type 2 and $\mathfrak{f}_0 = h(G) = (H \in \mathfrak{E} : Q(H) \cap (G) = \emptyset)$, which is a saturated formation with canonical local definition $F_0(p) = \mathfrak{f}_0$, for every prime $p \in \mathrm{char}\ \mathfrak{f}_0 = \mathbb{P}$. Consider now $\mathfrak{f} = (\mathfrak{f}_0)_s = (H \in \mathfrak{E} : s(H) \subseteq \mathfrak{f}_0)$ the unique largest subgroup-closed class contained in $\mathfrak{f}_0$. If we take into account the characterization obtained in [1, Th. (3.4)] it is not difficult to prove that $\mathfrak{f}$ is a saturated formation with canonical local definition $F(p) = \mathfrak{f}$, for every prime $p \in \mathrm{char}\ \mathfrak{f} = \mathbb{P}$. Moreover $\mathfrak{f}$ satisfies (ii)(a, b) and $\mathrm{Crit_s}(\mathfrak{f}) \cap b(\mathfrak{f})$ does not contain non-abelian simple groups, but $G \in \mathrm{Crit_s}(\mathfrak{f})$ and G is not biprimary.

It is clear that if $\mathfrak{f}$ is a subgroup-closed saturated formation such that $|\mathrm{char}\ \mathfrak{f}| = 2$, then $\mathfrak{f}$ has the property (α). Example 2 in [2] shows that there exist subgroup-closed saturated formations whose characteristic is composed of two prime numbers and for which the Shemetkov property does not hold.

In the next example we exhibit a subgroup-closed saturated formation $\mathfrak{f}$ such that $\mathfrak{f}$ has the property (α), $|\mathrm{char}\ \mathfrak{f}| \geq 3$, and $\mathfrak{f}$ does not have the Shemetkov property.

EXAMPLE 3. Let π be a set composed of odd primes such that $|\pi| \geq 3$. Consider the saturated formation $\mathfrak{f} = (\mathfrak{E}_{\pi} \cap \mathfrak{E}_{p'})\mathfrak{S}_p(\mathfrak{E}_{\pi} \cap \mathfrak{E}_{p'})$ of all π-groups of p-length at most one for some fixed prime $p \in \pi$. $\mathfrak{f}$ is subgroup-closed and char $\mathfrak{f} = \pi$. It is clear that $F(p) = \mathfrak{S}_p(\mathfrak{E}_{\pi} \cap \mathfrak{E}_{p'})$, $F(q) = \mathfrak{f}$ for $q \in \pi \setminus \{p\}$, and $F(q) = \emptyset$ otherwise. Therefore $\mathfrak{f} \cap \mathrm{Crit}_s(F(q)) = \emptyset$ for each $q \in \pi \setminus \{p\}$. Next we see that the condition $\mathfrak{f} \cap \mathrm{Crit}_s(F(p)) \subseteq \mathscr{P}(p)$ holds. Assume this is not true and let G be a group in $\mathfrak{f} \cap \mathrm{Crit}_s(F(p)) \setminus \mathscr{P}(p)$ of minimal order. Then G is a soluble π-group. Moreover G is primitive and $N = \mathrm{Soc}(G)$ is a q-group for some prime $q \in \pi \setminus \{p\}$. Let M be a core-free maximal subgroup of G. Since G is not in $F(p)$, we have that p divides $|M|$. Let $1 \neq P$ be a Sylow p-subgroup of M. Since $C_G(N) = N$ and $G \in \mathrm{Crit}_s(F(p))$, it follows that $G = NP \in \mathscr{P}(p)$. Consequently $\mathfrak{f} \cap \mathrm{Crit}_s(F(r)) \subseteq \mathscr{P}(r)$ for each prime $r \in \pi$. Moreover since every $\mathfrak{f}$-critical group is soluble, we have that $\mathrm{Crit}_s(\mathfrak{f}) \cap b(\mathfrak{f})$ does not contain primitive groups of type 2. Consequently $\mathfrak{f}$ satisfies (ii) of Theorem 4 and $\mathfrak{f}$ has the property (α). Nevertheless, $\mathfrak{f}$ does not have the Shemetkov property. To see this, let A be a cyclic group of order p. Then A has an irreducible and faithful module V over $GF(q)$. Let $B = [V]A$ be the corresponding semidirect product. Let W be an irreducible and faithful B-module over $GF(p)$ and construct the semidirect product $C = [W]B$. Then C is an $\mathfrak{f}$-critical group which is neither a Schmidt group nor a cyclic group of prime order.

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